Noise in hysteretic systems and stochastic processes on graphs

M. I. Freidlin,[1] I. D. Mayergoyz,[2] and R. Pfeiffer[1]
[1]Department of Mathematics, University of Maryland, College Park, Maryland 20742
[2]Department of Electrical and Computer Engineering, University of Maryland, College Park, Maryland 20742
(Received 5 January 2000)

It is shown that the theory of stochastic diffusion processes on graphs is a natural tool for the analysis of noise in hysteretic systems. In particular, by using this theory, analytical expressions for stationary characteristics of random outputs of some hysteretic systems are derived.

PACS number(s): 02.50.Ey, 02.50.Fz, 02.50.Ga

I. INTRODUCTION

It is well-known that hysteretic systems exhibit random spontaneous switchings. These switchings are usually attributed to the presence of internal noise as well as to the multiplicity of metastable states in hysteretic systems. Thus, outputs of hysteretic systems are random processes that can be viewed as hysteretic transformations of internal noise processes. Since hysteretic systems are endowed with memory, the random outputs are not Markovian processes even if random inputs are simplest Markovian processes. However, by considering these outputs jointly with internal noise processes, one sometimes arrives at multicomponent Markovian processes. These Markovian processes are defined on graphs. For this reason, the theory of stochastic processes on graphs is an appropriate tool for the analysis of noise in hysteretic systems. The theory of stochastic processes on graphs has been only recently developed [1] and applied to the study of random perturbations of Hamiltonian dynamical systems [2]. The main contribution of the paper is to demonstrate that the mathematical machinery of this theory is naturally suitable for the analysis of random outputs processes of certain hysteretic systems. As a by-product of this demonstration, we derive analytical expressions for stationary characteristics of these processes. These expressions are of interest in their own right.

The study of noise in hysteretic systems is of direct relevance to the modeling of thermal relaxations in hysteretic materials and systems. In magnetism, these thermally activated relaxations are commonly called ‘‘aftereffect’’ or ‘‘viscosity,’’ while in the area of superconductivity they are known as ‘‘creep.’’ The essence of these thermal relaxations is gradual (slow) temporal variations of output variables (for instance, magnetization) at constant in time external conditions (magnetic fields). These gradual temporal output variations are driven by internal thermal noise processes. For this reason, these output variations can be viewed (and modelled) as random output processes of hysteretic systems. The thermally activated relaxations are becoming increasingly important in magnetic data storage technology, where these relaxations corrupt (over time) the recorded information and, in this way, negatively affect the long-time reliability of data storage.

It has been previously suggested [3] to use the Preisach model of hysteresis driven with a stochastic input as a model for thermal relaxations in hysteretic systems. The Preisach model is constructed as a‘‘weighted superposition’’ of rectangular loop operators that are individually ‘‘driven’’ by the same thermal noise process. In the paper, the random output processes of rectangular loop operators are first analyzed and then this analysis is applied to the study of random output processes of the Preisach model.

II. TECHNICAL DISCUSSION

To start the discussion, consider hysteretic nonlinearities, which are represented by rectangular loop operators $\hat{\gamma}_{\alpha\beta}$ (see Fig. 1). The rectangular loop operator is mathematically defined as follows:

$$i_t = \hat{\gamma}_{\alpha\beta} x_t, \tag{1}$$

$$i_t = \begin{cases} 1 & \text{if } x_t > \alpha, \\ -1 & \text{if } x_t < \beta, \\ 1 & \text{if } x_t \epsilon(\beta,\alpha) \text{ and } x_{\tau_{(t)}} = \alpha, \\ -1 & \text{if } x_t \epsilon(\beta,\alpha) \text{ and } x_{\tau_{(t)}} = \beta, \end{cases} \tag{2}$$

where $\tau(t)$ is the value of time at which the last threshold (α or β) was attained.

It is apparent that $\hat{\gamma}_{\alpha\beta}$ are the simplest hysteretic operators. Nevertheless, they are used as the main building blocks for complicated and more or less realistic hysteresis models such as the Preisach models [3,4]. For this reason, certain results obtained for $\hat{\gamma}_{\alpha\beta}$ operators are directly relevant to the Preisach models. In addition, random switchings of $\hat{\gamma}_{\alpha\beta}$ op-

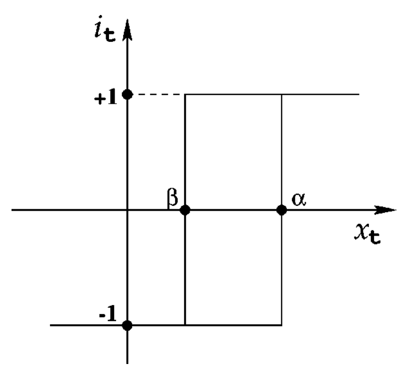

FIG. 1. Rectangular hysteresis loop that represents the operator $\hat{\gamma}_{\alpha\beta} x_t$.

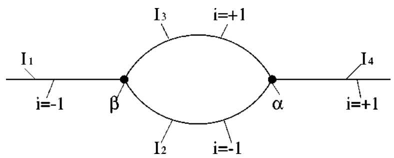

FIG. 2. Four edge graph on which two component process $\vec{y}_t$ is defined.

erators may be of interest in their own right in those areas of information technology where rectangular loop nonlinearities are used for data storage.

In the sequel, a noise process x_t is assumed to be described by the Ito stochastic differential equation:

$$dx_t = b(x_t)dt + \sigma(x_t)dW_t. \tag{3}$$

where W_t is the Wiener process, while b and σ are known functions.

This noise leads to random switching of the rectangular loop operator $\hat{\gamma}_{\alpha\beta}$. Thus, the output i_t is a random binary process. This process is not Markovian. However, the two component process $\vec{y}_t = (i_t, x_t)$ is Markovian. This is because the rectangular loop operators describe hysteresis with local memory. This means that joint specifications of current values of input and output uniquely define the states of this hysteresis.

The two component process $\vec{y}_t$ is defined on the four edge graph shown in Fig. 2. The binary process i_t assumes constant values on each edge I_k of the above graph. This justifies the following concise notation for the transition probability density:

$$\rho(t, \vec{y}|\vec{y}_0)|_{\vec{y}\in I_k} = \rho^{(k)}(t, x|\vec{y}_0). \tag{4}$$

It is obvious that

$$\begin{gathered} \rho^{(1)} = \rho \quad \text{for } x \leqslant \beta, \quad \rho^{(4)} = \rho \quad \text{for } x \geqslant \alpha, \\ \rho^{(2)} + \rho^{(3)} = \rho \quad \text{for } x \in [\beta, \alpha], \end{gathered} \tag{5}$$

where ρ is the transition probability density of the process x_t, which is assumed to be known.

According to the theory of Markovian processes, the following equality is valid for $\rho^{(k)}$:

$$\sum_{k=1}^{4} \int_{I_k} f \frac{\partial \rho^{(k)}}{\partial t} dx = \sum_{k=1}^{4} \int_{I_k} (\hat{L} f) \rho^{(k)} dx. \tag{6}$$

Here,

$$\hat{L} = \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} + b(x) \frac{\partial}{\partial x}$$

is the generator of the semigroup of the process x_t, while f is a function that is continuous on the entire graph and sufficiently smooth inside the edges and that satisfies certain ''gluing'' (interface) conditions at α and β. These interface conditions follow from the Markovian nature of the process $\vec{y}_t$ on the entire graph [1]. In our case, the process $\vec{y}_t$ ''spends zero time'' at the graph vertices. In this situation, the interface conditions can be written as follows [1]:

$$\sum_k \lambda_{kj} \frac{df_{I_k}}{dx}\bigg|_{O_j} = 0, \quad \lambda_{kj} \geqslant 0. \tag{7}$$

Here, $f_{I_k} = f|_{I_k}$, summation is performed over all edges connected to a graph vertex O_j, while the derivatives are taken along the edges in outward directions with respect to O_j. It is known [1] that constants λ_{kj} are (roughly speaking) proportional to the probabilities that the process will ''move'' from vertex O_j along the edges I_k. It is clear that in our case there is zero probability that the process $\vec{y}_t$ will move from the vertex β along the edge I_3, while the random motions along the edges I_1 and I_2 are equally probable. The similar assertion is valid for the vertex α. As a result, we arrive at the following interface conditions:

$$\frac{\partial f_{I_1}}{dx}\bigg|_{\beta} = \frac{df_{I_2}}{dx}\bigg|_{\beta}, \quad \frac{df_{I_3}}{dx}\bigg|_{\alpha} = \frac{df_{I_4}}{dx}\bigg|_{\alpha}, \tag{8}$$

while the values of the derivatives $df_{I_3}/dx|_{\beta}$ and $df_{I_2}/dx|_{\alpha}$ are entirely arbitrary. It is understood that differentiation in (8) is performed in the direction of increasing values of x.

By integrating by parts in the equality (6) and by taking into account the interface conditions (8) and the choice inherent in f, $df_{I_3}/dx|_{\beta}$ and $df_{I_2}/dx|_{\alpha}$, one finds that the transition probability density $\rho^{(3)}$ satisfies the forward Kolmogorov equation

$$\frac{\partial \rho^{(3)}}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(x) \rho^{(3)}) - \frac{\partial}{\partial x} (b(x) \rho^{(3)}) \tag{9}$$

and the following boundary conditions:

$$\rho^{(3)}|_{\beta} = 0, \quad \rho^{(3)}|_{\alpha} = \rho|_{\alpha}. \tag{10}$$

It is also tacitly understood that the standard δ-type initial condition is imposed at $\vec{y}_0$.

A similar initial boundary value problem can be stated for $\rho^{(2)}$. However, $\rho^{(2)}$ can be also found by using formula (5).

The solution to the initial-boundary value problem (9)–(10) can be found in terms of parabolic cylinder functions and their Laplace transforms in the case when x_t is the Ornstein–Uhlenbeck process $(dx_t = -b(x_t - x_0)dt + \sigma\, dW_t)$ with expected value x_0. The Ornstein–Uhlenbeck process is very appealing as a noise model because of its stationary and Gaussian nature.

Simpler analytical results can be obtained for stationary densities $\rho_{st}^{(3)}$ and $\rho_{st}^{(2)}$. In this case, we have to deal with the following boundary value problem for the ordinary differential equation

$$\frac{1}{2} \frac{d^2}{dx^2} (\sigma^2(x) \rho_{st}^{(3)}) - \frac{d}{dx} (b(x) \rho_{st}^{(3)}) = 0, \tag{11}$$

$$\rho_{st}^{(3)}(\beta) = 0, \quad \rho_{st}^{(3)}(\alpha) = \rho_{st}(\alpha). \tag{12}$$

Although the analytical solution to the above boundary value problem can be written out for any stationary diffusion process x_t, below we present this solution only for the case of the Ornstein–Uhlenbeck process:

$$\rho_{st}^{(3)}(x)=\rho_{st}(x)\varphi(x,\alpha,\beta),$$
$$\rho_{st}^{(2)}(x)=\rho_{st}(x)[1-\varphi(x,\alpha,\beta)], \tag{13}$$

where

$$\rho_{st}(x)=\sqrt{\frac{b}{\pi\sigma^2}}e^{-b(x-x_0)^2/\sigma^2}, \tag{14}$$

$$\varphi(x,\alpha,\beta)=\frac{\int_\beta^x e^{\beta(y-x_0)^2/\sigma^2}\,dy}{\int_\beta^\alpha e^{\beta(y-x_0)^2/\sigma^2}\,dy}. \tag{15}$$

If we consider the probability current

$$J_k(x)=-\frac{\sigma^2}{2}\frac{d\rho^{(k)}(x)}{dx}-b(x-x_0)\rho^{(k)}(x),$$

then it is easy to conclude from formulas (13), (14), and (15) that $J_1(x)=J_4(x)\equiv 0$, while $J_2(x)=-J_3(x)\neq 0$. Thus, there exists a nonzero probability current circulating in the loop formed by edges I_3 and I_2. The existence of the circulating loop current can be considered as the manifestation of the lack of detailed balance in the two component process $\vec{y}_t$. The existence of this circulating current can be also traced to energy losses associated with the random switchings of rectangular loop $\hat{\gamma}_{\alpha\beta}$. This dissipated energy is extracted from noise, which is the only source of energy present in the discussed problem. The situation here is analogous to one observed for stochastic resonance, where a feeble deterministic signal alone cannot affect switchings. These switchings are assisted by the internal noise and they are accompanied by the extraction of energy from the noise [5].

It is instructive to compute the expected value $\overline{i_t}$ of the binary output process and its variance $\sigma_{i_t}^2$. It is clear that

$$\overline{i_t}=E_{st}(\hat{\gamma}_{\alpha\beta}x_t)=P_{st}^{\alpha\beta}(i_t=1)-P_{st}^{\alpha\beta}(i_t=-1), \tag{16}$$

which is equivalent to

$$\overline{i_t}=2P_{st}^{\alpha\beta}(i_t=1)-1. \tag{17}$$

It is apparent that

$$P_{st}^{\alpha\beta}(i_t=1)=\int_\beta^\alpha \rho_{st}^{(3)}(x)dx+\int_\alpha^\infty \rho_{st}(x)dx, \tag{18}$$

which leads to

$$\overline{i_t}=2\left[\int_\beta^\alpha \rho_{st}^{(3)}(x)dx+\int_\alpha^\infty \rho_{st}(x)dx\right]-1. \tag{19}$$

When $\overline{i_t}$ is computed, $\sigma_{i_t}^2$ can be calculated as follows:

$$\sigma_{i_t}^2=1-\overline{i_t}^2. \tag{20}$$

Calculations are substantially facilitated by the observation that

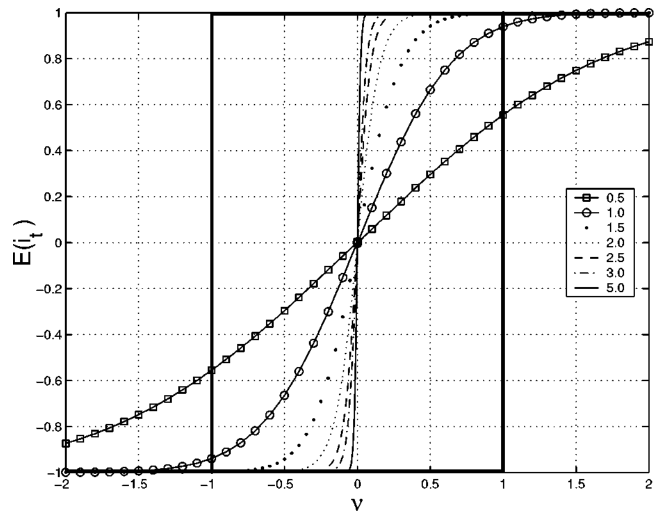

FIG. 3. Expected values of i_t as functions of the normalized expected value (normalized bias) of x_t for various normalized threshold values of $\tilde{\alpha}$.

$$\int_0^x e^{y^2}\,dy=\frac{\sqrt{\pi}}{2}\,\mathrm{erfi}(x), \tag{21}$$

$$\int e^{-x^2}\,\mathrm{erfi}(x)dx=\frac{x^2}{\sqrt{\pi}}\,{}_2F_2(1,1;\tfrac{3}{2},2;-x^2), \tag{22}$$

where $\mathrm{erfi}(x)$ and ${}_2F_2$ are ''imaginary error function'' and ''generalized hypergeometric function,'' respectively.

By using the formulas presented above, the expected value $\overline{i_t}$ and variance $\sigma_{i_t}^2$ have been computed as functions of the expected (''bias'') value x_0 of the input process x_t for the case of hysteretic nonlinearities represented by symmetric rectangular loops $\hat{\gamma}_{\alpha,-\alpha}$. The results of calculations are shown in Figs. 3 and 4, respectively. These results are plotted for normalized values of input bias $\nu=x_0/\alpha$ and normalized values of switching thresholds $\tilde{\alpha}=\alpha/\lambda$, where $\lambda^2=\sigma^2/\beta$ is the variance of the stationary distribution of the input Ornstein–Uhlenbeck process x_t. The dependence of $\overline{i_t}$

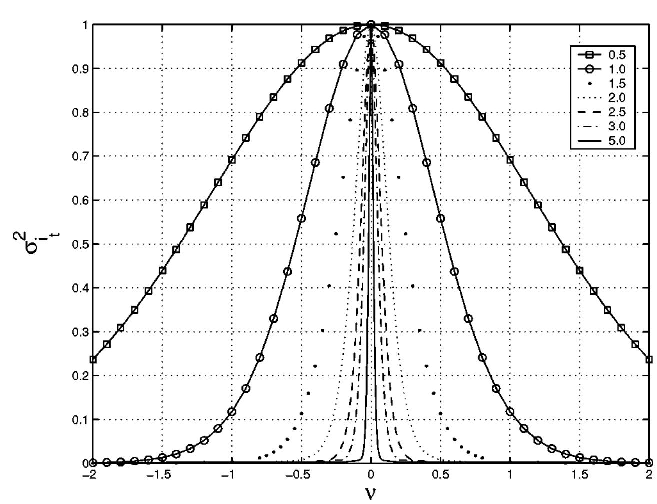

FIG. 4. Variance of i_t as functions of the normalized bias of x_t for various normalized threshold values of $\tilde{\alpha}$.

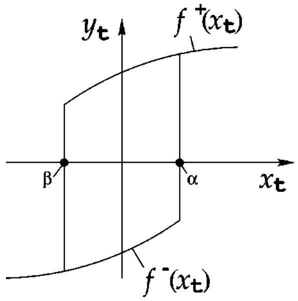

FIG. 5. Hysteresis loop with ''curved'' ascending and descending branches.

on x_0 can be interpreted in magnetics as ''anhysteretic'' magnetization curve [6]. This anhysteretic curve depends on the noise variance. This dependence is especially appreciable when the noise standard deviation λ is comparable with the switching threshold value α.

Next, we shall apply the obtained results to the case of hysteresis loop with ''curved'' ascending and descending branches (Fig. 5). This type of hysteresis loop is exhibited, for instance, by Stoner–Wohlfarth particles [6,7] when the applied magnetic field is restricted to vary along one direction. Suppose that the loop shown in Fig. 5 is driven by the Ornstein–Uhlenbeck process x_t and we are interested in the stationary distribution of the output process y_t. The process y_t is not a binary one. However, it admits the following representation in terms of the binary process $i_t=\hat{\gamma}_{\alpha\beta}x_t$:

$$y_t=\frac{f^+(x_t)-f^-(x_t)}{2}i_t+\frac{f^+(x_t)+f^-(x_t)}{2}, \tag{23}$$

where the meaning of $f^+(x_t)$ and $f^-(x_t)$ is clear from Fig. 5.

By using formula (23) and the appropriate change of variables, we obtain the following expression for the stationary distribution density $\xi^+_{st}(y)$ of the process y_t along the descending branch:

$$\xi^+_{st}(y)=\rho^{(3)}_{st}(g^+(y))\frac{dg^+(y)}{dy}. \tag{24}$$

Here, $g^+(y)$ is the inverse of the function $y=f^+(x)$. The last formula is valid for $\beta<x<\alpha$. For $x>\alpha$, density $\rho^{(3)}_{st}$ should be replaced by ρ_{st}. In a similar way, the density $\xi^-_{st}(y)$ along the ascending branch can be computed.

The results obtained for $\hat{\gamma}_{\alpha\beta}$ operators can also be used to compute the stationary characteristics of the output process f_t of the Preisach model driven by the stochastic process x_t. The Preisach model of hysteresis is constructed as a ''weighted superposition'' of rectangular loop operators [3,4].

$$f_t=\int\int_{\alpha\geqslant\beta}\mu(\alpha,\beta)\hat{\gamma}_{\alpha\beta}x_t\,d\alpha\,d\beta. \tag{25}$$

It has been shown [4] that the ''wiping-out property'' and the ''congruency of minor loops'' constitute the necessary and sufficient conditions for the representation of actual hysteresis nonlinearities by the Preisach model. In this way, the Preisach model has been established as a legitimate mathematical tool [8,9]. At the same time, the above necessary and sufficient conditions have clearly revealed the limits of applicability of the Preisach model as well as its physical universality within those limits. As a result, the Preisach model has been used for the description of hysteresis of various physical nature such as superconducting hysteresis [4,10], mechanical hysteresis of consolidated granular materials [11–13], hysteresis of shape-memory alloys [14] and piezoceramics [15]. Connections between the Preisach model and hysteresis due to random structural disorder have also been established [16].

The Preisach model driven with a stochastic input is an effective model for thermal relaxations in hysteretic systems. For this reason, stochastic characteristics of the output of this model are of interest.

The Preisach model describes hysteresis with nonlocal memories [3,4]. For this reason, the output process f_t cannot be embedded as a component of some Markovian process. Nevertheless, the moments of f_t can be computed.

From formula (25), we find the following expression for the stationary expected value of f_t:

$$\overline{f^{st}_t}=\int\int_{\alpha\geqslant\beta}\mu(\alpha,\beta)E_{st}(\hat{\gamma}_{\alpha\beta}x_t)\,d\alpha\,d\beta, \tag{26}$$

where $E_{st}(\hat{\gamma}_{\alpha\beta}x_t)$ can be evaluated by using expressions (13)–(22). If it is desired to evaluate the stationary value of the second moment $E_{st}(f^2_t)$ of the output process f_t and its variance, the following integral must be evaluated:

$$\begin{aligned}E_{st}(f^2_t)&\\&=\int\int_{\alpha_1\geqslant\beta_1}\int\int_{\alpha_2\geqslant\beta_2}E_{st}(\hat{\gamma}_{\alpha_1\beta_1}x_t\hat{\gamma}_{\alpha_2\beta_2}x_t)\\&\quad\times\mu(\alpha_1,\beta_1)\mu(\alpha_2,\beta_2)\,d\alpha_1\,d\beta_1\,d\alpha_2\,d\beta_2,\end{aligned} \tag{27}$$

where $E_{st}(\cdot)$ stands for stationary expected value.

To compute $E_{st}(\hat{\gamma}_{\alpha_1\beta_1}x_t\hat{\gamma}_{\alpha_2\beta_2}x_t)$, we consider the three component Markovian process $\vec{z}_t=(i^1_t,i^2_t,x_t)$, where $i^1_t=\hat{\gamma}_{\alpha_1\beta_1}x_t$ and $i^2_t=\hat{\gamma}_{\alpha_2\beta_2}x_t$. Depending on the relation between α_1, β_1, α_2, and β_2, this process is defined on graphs shown in Fig. 6. By using the same line of reasoning as before, one can easily arrive at the following expressions for the stationary densities $\rho_{st}(\vec{z}_t)|_{I_k}=\rho^{(k)}_{st}(x)$.

In the first case [Fig. 6(a)] when two rectangular loops do not overlap $(\beta_1<\alpha_1<\beta_2<\alpha_2)$, we have

$$\rho^{(k)}_{st}=\rho_{st}\quad\text{for }k=1,4,7,\quad\rho^{(3)}_{st}(x)=\rho_{st}(x)\varphi(x,\alpha_1,\beta_1), \tag{28}$$

$$\rho^{(6)}_{st}(x)=\rho_{st}(x)\varphi(x,\alpha_2,\beta_2),\quad\rho^{(2)}_{st}=\rho_{st}-\rho^{(3)}_{st},$$
$$\rho^{(5)}_{st}=\rho_{st}-\rho^{(6)}_{st}. \tag{29}$$

In the second case [Fig. 6(b)] when two rectangular loops completely overlap $(\beta_1<\beta_2<\alpha_2<\alpha_1)$, we have

$$\rho^{(k)}_{st}=\rho_{st}\quad\text{for }k=1,8,\quad\rho^{(3)}_{st}(x)=\rho_{st}(x)\varphi(x,\alpha_1,\beta_1), \tag{30}$$

$$\rho^{(2)}_{st}=\rho_{st}-\rho^{(3)}_{st},\quad\rho^{(5)}_{st}(x)=\rho^{(3)}_{st}(x)\varphi(x,\alpha_2,\beta_2), \tag{31}$$

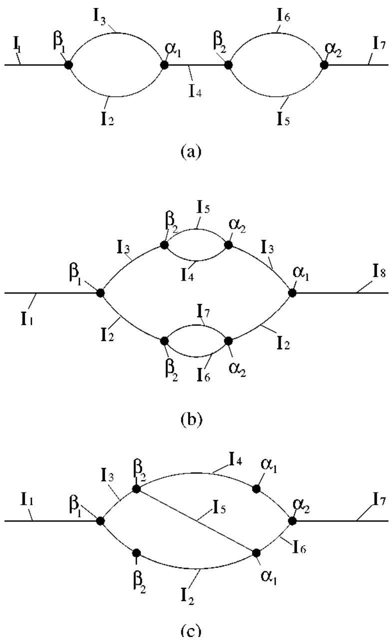

FIG. 6. Graphs on which three component process $\vec{z}_t$ is defined.

$$\rho_{st}^{(4)}=\rho_{st}^{(3)}-\rho_{st}^{(5)}, \quad \rho_{st}^{(7)}(x)=\rho_{st}^{(2)}(x)\varphi(x,\alpha_2,\beta_2),$$

$$\rho_{st}^{(6)}=\rho_{st}^{(2)}-\rho_{st}^{(7)}. \tag{32}$$

Finally, in the third case [Fig. 6(c)] when two rectangular loops partially overlap ($\beta_1<\beta_2<\alpha_1<\alpha_2$), we have

$$\rho_{st}^{(k)}=\rho_{st} \quad \text{for } k=1,7, \quad \rho_{st}^{(2)}(x)=\rho_{st}(x)[1-\varphi(x,\alpha_1,\beta_1)], \tag{33}$$

$$\rho_{st}^{(3)}=\rho_{st}-\rho_{st}^{(2)}, \quad \rho_{st}^{(4)}(x)=\rho_{st}(x)\varphi(x,\alpha_2,\beta_2), \tag{34}$$

$$\rho_{st}^{(5)}=\rho_{st}-\rho_{st}^{(2)}-\rho_{st}^{(4)}, \quad \rho_{st}^{(6)}=\rho_{st}-\rho_{st}^{(4)}. \tag{35}$$

It is worthwhile noting that in the last case there is no graph edge corresponding to $i_t^1=-1$ and $i_t^2=+1$ because these simultaneous values of i_t^1 and i_t^2 are not consistent with the definition of rectangular loops operators $\hat{\gamma}_{\alpha_1\beta_1}$ and $\hat{\gamma}_{\alpha_2\beta_2}$. By using the above expressions for $\rho_{st}^{(k)}, E_{st}(\hat{\gamma}_{\alpha_1\beta_1}x_t\hat{\gamma}_{\alpha_2\beta_2}x_t)$ can be computed.

As an example, consider a particular case of the Preisach model when all rectangular $\hat{\gamma}$-loops are symmetric: $\hat{\gamma}_{\alpha,-\alpha}x_t=\hat{\gamma}_\alpha x_t$. Mathematically, this case is obtained when the ''weight'' function $\mu(\alpha,\beta)$ has the form

$$\mu(\alpha,\beta)=\xi(\alpha)\delta(\alpha+\beta), \tag{36}$$

and the Preisach model (25) is reduced to

$$f_t=\int_0^{\alpha_0}\xi(\alpha)\hat{\gamma}_\alpha x_t\,d\alpha. \tag{37}$$

For many magnetic materials, the weight function $\mu(\alpha,\beta)$ is usually narrowly peaked around the line $\alpha=-\beta$. For these materials, formulas (36) and (37) can be regarded as fairly good approximations. At the same time, the calculations are considerably simplified because, in the case of symmetric loops, any two loops completely overlap. As a result, one has to deal with the three component Markovian process $\vec{z}_t$ defined only on the graphs shown in Fig. 6(b).

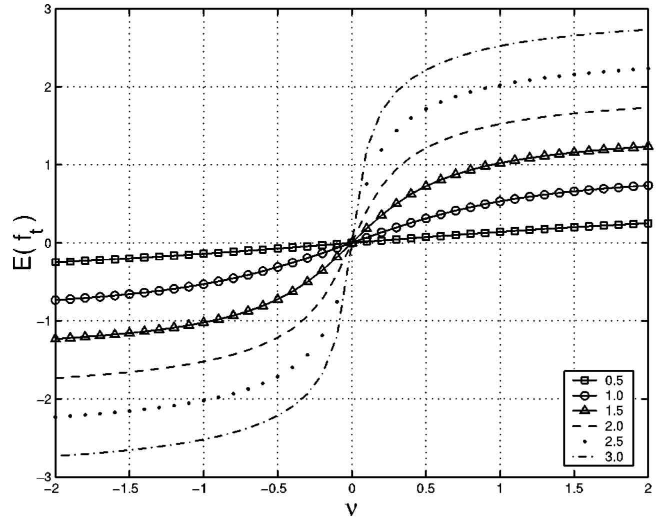

FIG. 7. Expected values of f_t as functions of the normalized bias of x_t for various $\tilde{\alpha}_0$.

For the case of model (37), formulas (26) and (27) can be written as follows:

$$\overline{f_t^{st}}=\int_0^{\alpha_0}\xi(\alpha)E_{st}(\hat{\gamma}_\alpha x_t)d\alpha, \tag{38}$$

$$E_{st}(f_t^2)=2\int_0^{\alpha_0}\xi(\alpha)\left(\int_0^{\alpha}\xi(\alpha')E_{st}(\hat{\gamma}_\alpha x_t\hat{\gamma}_{\alpha'}x_t)d\alpha'\right)d\alpha. \tag{39}$$

To compute $E_{st}(\hat{\gamma}_\alpha x_t)$, formulas (13)–(22) can be used. To evaluate $E_{st}(\hat{\gamma}_\alpha x_t\hat{\gamma}_{\alpha'}x_t)$ in (39), we first remark that

$$E_{st}(\hat{\gamma}_\alpha x_t\hat{\gamma}_{\alpha'}x_t)=2P(\hat{\gamma}_\alpha x_t\hat{\gamma}_{\alpha'}x_t=1)-1. \tag{40}$$

To find $P(\hat{\gamma}_\alpha x_t\hat{\gamma}_{\alpha'}x_t)$, one has to integrate the appropriate $\rho_{st}^{(k)}(x)$ over those edges of the graph shown in Fig. 6(b) on which $\hat{\gamma}_\alpha x_t$ and $\hat{\gamma}_{\alpha'}x_t$ have the same signs. This leads to the formula:

$$E_{st}(\hat{\gamma}_\alpha x_t\hat{\gamma}_{\alpha'}x_t)=2\Bigg[\int_{-\alpha'}^{\alpha'}\rho_{st}^{(5)}(x)dx+\int_{\alpha'}^{\alpha}\rho_{st}^{(3)}(x)dx+\int_{\alpha}^{\infty}\rho_{st}(x)dx+\int_{-\alpha'}^{\alpha'}\rho_{st}^{(6)}(x)dx+\int_{-\alpha}^{-\alpha'}\rho_{st}^{(2)}(x)dx+\int_{-\infty}^{-\alpha}\rho_{st}(x)dx\Bigg]-1. \tag{41}$$

The integrals in the last expression can be evaluated by using formulas (30)–(32) and taking advantage of relations (21) and (22). Some sample results of calculations are shown in Figs. 7 and 8 for $\overline{f_t^{st}}$ and $E_{st}(f_t^2)$, respectively. In these calculations, it was assumed that $\xi(\alpha)=1$ and $\overline{f_t^{st}}$ and $E_{st}(f_t^2)$ were computed as functions of normalized values of input bias $\nu=x_0/\alpha_0$ for various normalized values of $\tilde{\alpha}_0=\alpha_0/\lambda$, where λ^2 is the variance of the stationary distribution of x_t.

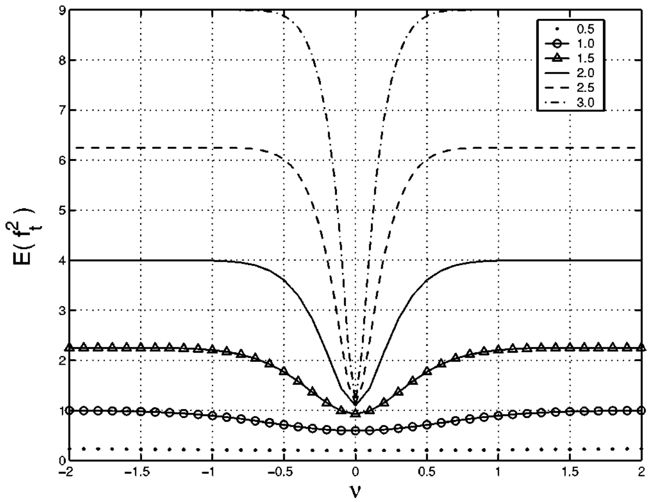

FIG. 8. Second moment of f_t as functions of the normalized bias of x_t for various $\tilde{\alpha}_0$.

The values of $\overline{f_t^{st}}$ and $E_{st}(f_t^2)$ have been also normalized by λ and λ^2, respectively. It is worthwhile to note that, as evident from Fig. 8, the asymptotic values of $E_{st}(f_t^2)$ are equal to $\tilde{\alpha}_0^2$ and coincide with asymptotic values of $(\overline{f_t^{st}})^2$. This guarantees zero asymptotic values for variance $\sigma_{f_t}^2$.

The described formalism of stochastic processes on graphs can be further extended to compute higher order moments of the output process f_t. This extension is more or less straightforward in the case of the model (37). In this case, the relevant multicomponent Markovian processes are defined on the graph shown in Fig. 9 and, by using the same line of reasoning as before, we can derive the following explicit expression for $\rho_{st}^{(2k+1)}(x)$ for edges I_{2k+1}:

$$\rho_{st}^{(2k+1)}(x)=\rho_{st}(x)\prod_{j=1}^{k}\varphi(x,\alpha_j,-\alpha_j). \tag{42}$$

Similar expressions can be derived for other graph edges.

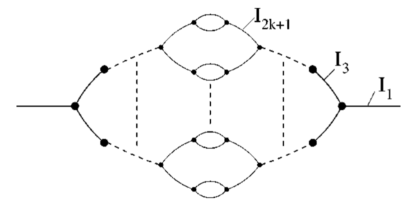

FIG. 9. Generic graph on which multicomponent Markovian processes are defined.

III. CONCLUSION

The analysis of random output processes of rectangular loop operators driven by diffusion processes is first presented. It is observed that two component (output–input) processes are Markovian and they are defined on graphs. By using the theory of stochastic processes on graphs, initial-boundary value problems are derived for the transitional probability density of two component (output–input) processes. By employing these boundary value problems, analytical expressions are found for stationary probability densities of the two component processes. These results are extended to the case of hysteresis loops with ''curved'' ascending and descending branches, which are typical for Stoner–Wohlfarth particles. The results obtained for the rectangular loop operators are applied to the analysis of random output processes of the Preisach model driven with stochastic inputs. This analysis is of physical significance because the Preisach model driven with stochastic inputs is an effective model for thermally activated relaxations in hysteretic materials and systems.

ACKNOWLEDGMENTS

The authors are grateful to Mr. M. Huang for his valuable assistance in calculations. The reported research has been supported by the U.S. Department of Energy, Engineering Research Program.

[1] M. I. Freidlin, A. D. Wentzell, Ann. Prob. **21**, 2215 (1993).
[2] M. I. Freidlin, *Markov Processes and Differential Equations: Asymptotic Problems* (Birkhauser-Verlag, Berlin, 1996).
[3] I. D. Mayergoyz, *Mathematical Models of Hysteresis* (Springer-Verlag, New York, 1991).
[4] I. D. Mayergoyz, Phys. Rev. Lett. **56**, 1518 (1986).
[5] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[6] G. Bertotti, *Hysteresis in Magnetism* (Academic, Boston, 1998).
[7] E. C. Stoner and E. P. Wohlfarth, Philos. Trans. R. Soc. London, Ser. A **240**, 599 (1948).
[8] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions* (Springer-Verlag, New York, 1996).
[9] A. Visintin, *Differential Models of Hysteresis* (Springer-Verlag, New York, 1994).
[10] I. D. Mayergoyz and T. Keim, J. Appl. Phys. **67**, 5466 (1990).
[11] R. A. Guyer, K. R. McCall, and G. N. Boitnott, Phys. Rev. Lett. **74**, 3491 (1995).
[12] R. A. Guyer, J. TenCate, and P. A. Johnson, Phys. Rev. Lett. **82**, 3280 (1999).
[13] R. A. Guyer and P. A. Johnson, Phys. Today **52**, 30 (1999).
[14] K. Wilmanski, Int. J. Eng. Sci. **31**, 1121 (1993).
[15] P. Ge and M. Jouaneh, Precis. Eng. **17**, 211 (1995).
[16] G. Bertotti, I. D. Mayergoyz, V. Basso, and A. Magni, Phys. Rev. E **60**, 1428 (1999).